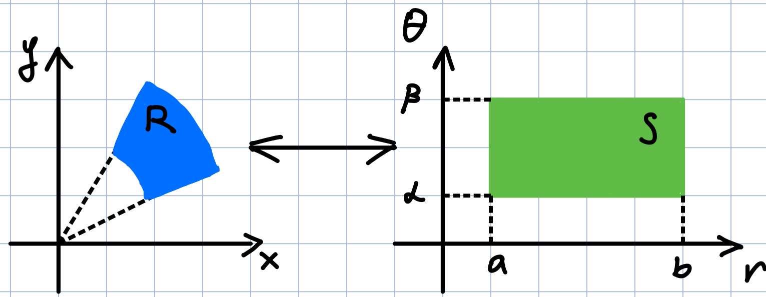
Change of variables in multiple integrals

in 1D integrals: $\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du = \int_c^d f(x(u))\frac{dx}{du}\,du$

substitution $x = g(u)$
with $a = g(c)$
$b = g(d)$

in 2D integrals:
(example)



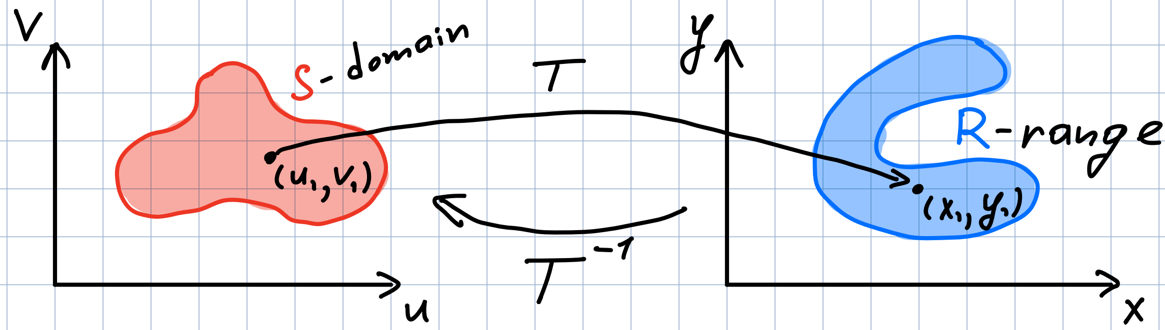
$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,\underline{\underline{r}}\,dr\,d\theta$$

region in $r\theta$-plane corresponding to R (region in xy-plane)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Generally: T - transformation from uv-plane to xy-plane, given by $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ $(*)$

- $T(u_1, v_1) = \underbrace{(x_1, y_1)}_{\text{image of } (u_1, v_1)}$
- R = image of S



- If no two points in S have same image, T is "one-to-one"

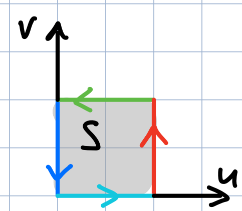
If T one-to-one, there is an inverse transformation: $\begin{cases} u = G(x,y) \\ v = H(x,y) \end{cases}$

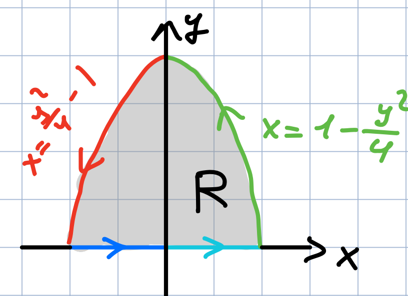
// from solving $(*)$ //

Ex: $T: (u,v) \longmapsto (x = u^2 - v^2,\ y = 2uv)$. Find the image R of the square S,

$$S = \left\{ (u,v) \,\middle|\, \begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le 1 \end{array} \right\}$$

Sol: T: boundary of S $\longrightarrow$ boundary of R





$(u,0) \longmapsto (u^2, 0)$ | $(u,1) \longmapsto (u^2-1, 2u)$

$(1,v) \longmapsto (1-v^2, 2v)$ | $(0,v) \longmapsto (-v^2, 0)$

$\begin{array}{l} x = 1 - v^2 \\ y = 2v \end{array} \implies \boxed{x = 1 - \left(\frac{y}{2}\right)^2}$ | $\begin{array}{l} x = u^2 - 1 \\ y = 2u \end{array} \implies \boxed{x = \left(\frac{y}{2}\right)^2 - 1}$

Jacobian of T:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

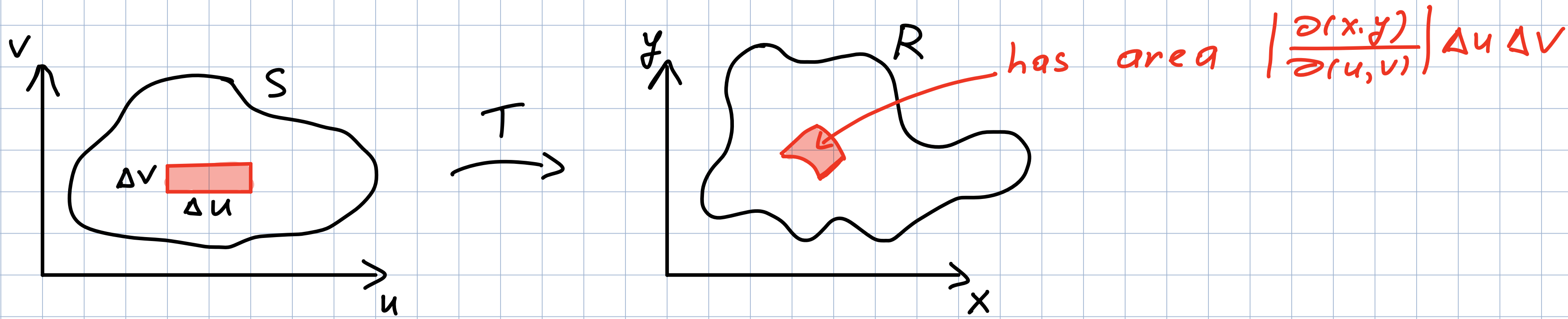
Change of variables in a double integral:

Suppose T maps S (in uv-plane) to R (in xy-plane) and is differentiable and 1-to-1. Suppose Jacobian $\neq 0$ everywhere inside S.

Then

$$\boxed{\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

Idea:



Ex: (Polar coordinates)

$T: (r,\theta) \longrightarrow (x,y)$

$x = r\cos\theta$
$y = r\sin\theta$

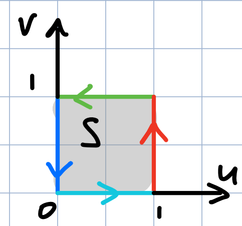
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \underbrace{\frac{\partial x}{\partial r}}_{\cos\theta}\underbrace{\frac{\partial y}{\partial \theta}}_{r\cos\theta} - \underbrace{\frac{\partial x}{\partial \theta}}_{-r\sin\theta}\underbrace{\frac{\partial y}{\partial r}}_{\sin\theta} = r \Rightarrow$$

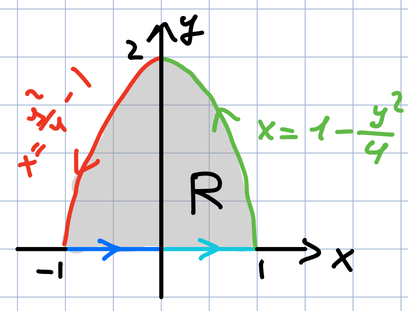
$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Ex: $T: (u,v) \longmapsto (x = u^2 - v^2,\ y = 2uv)$. Find the image R of the square S,

$$S = \left\{ (u,v) \,\middle|\, \begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le 1 \end{array} \right\}$$

Sol: T: boundary of $S \longrightarrow$ boundary of R





$(u,0) \longmapsto (u^2, 0)$ | $(u,1) \longmapsto (u^2-1, 2u)$

$(1,v) \longmapsto (1-v^2, 2v)$ | $(0,v) \longmapsto (-v^2, 0)$

$\begin{array}{l} x = 1 - v^2 \\ y = 2v \end{array} \Longrightarrow x = 1 - \left(\frac{y}{2}\right)^2$ | $\begin{array}{l} x = u^2 - 1 \\ y = 2u \end{array} \Longrightarrow x = \left(\frac{y}{2}\right)^2 - 1$

Ex: Find $\iint_R y \, dA$, where R is

Sol: Use the change of coordinates $\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$

Note that $R = T(S)$.

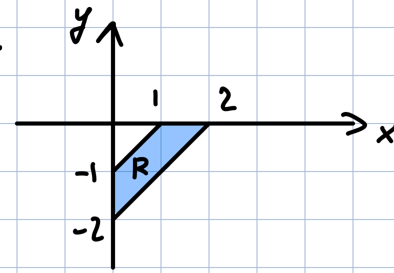
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0 \quad \text{in } S$$

So,

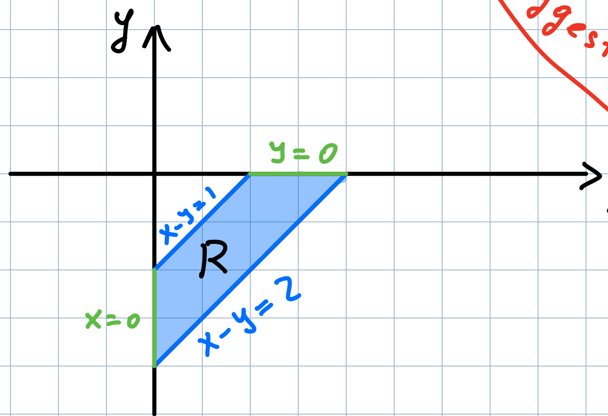
$$\iint_R y \, dA = \iint_S \underbrace{y}_{=2uv} (4u^2 + 4v^2) \, du \, dv = 8 \int_0^1 \int_0^1 (u^3 v + u v^3) \, du \, dv = 8 \int_0^1 \left(\frac{v}{4} + \frac{v^3}{2} \right) dv$$

$$= 8 \left(\frac{1}{8} + \frac{1}{8} \right) = 2$$

Ex: Find $I = \iint_R e^{\frac{x+y}{x-y}}\, dA$, where R is a trapezoid:



Sol:



suggested by

Make a change of variables:

$$\begin{cases} x+y=u \\ x-y=v \end{cases} \rightsquigarrow \quad x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}$$

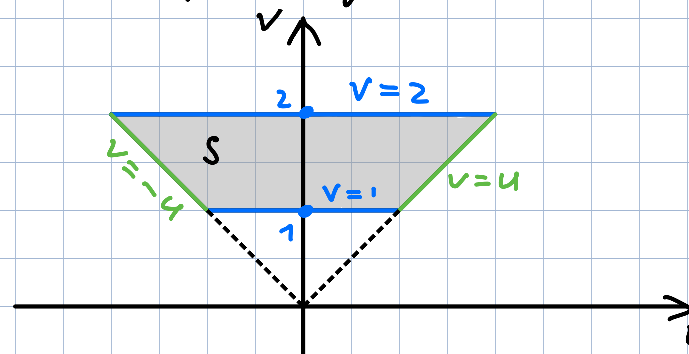
Region S in uv-plane corresponding to R:

$x-y=1 \longrightarrow v=1$

$x-y=2 \longrightarrow v=2$

$y=0 \longrightarrow v=u$

$x=0 \longrightarrow v=-u$



$$S = \left\{ (u,v) : \begin{array}{c} 1 \le v \le 2 \\ -v \le u \le v \end{array} \right\}$$

Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

So, $\boxed{I} = \iint_S e^{\frac{u}{v}} \left|-\frac{1}{2}\right| du\, dv = \frac{1}{2} \int_1^2 \underbrace{\int_{-v}^{v} e^{\frac{u}{v}}\, du}_{v e^{\frac{u}{v}}\Big|_{u=-v}^{u=v} = v(e-e^{-1})}\, dv$

$$= \frac{1}{2} \int_1^2 (e-e^{-1}) v\, dv = \frac{1}{2}(e-e^{-1}) \frac{v^2}{2}\Big|_{v=1}^{v=2} = \boxed{\frac{3}{4}\left(e - \frac{1}{e}\right)}$$